A REFINED FOUR-VARIABLE PLATE THEORY FOR THERMAL BUCKLING ANALYSIS OF FUNCTIONALLY GRADED SANDWICH PLATES

Hamidi Ahmed[12], Makani Abdelkadir[12], Bekkar Izzedine[12], and Zidour Mohamed[3]

[1]Department of Civil Engineering & hydraulics, Tahri Mohamed University, Béchar, Algeria
[2]EMIA Research Laboratory - Tahri Mohamed University
hamidi.ahmed@univ-bechar.dz
[3]Ibn khaldoun University Department of Civil Engineering,Tiaret, Algeria

ABSTRACT

In this paper, we have developed and investigated the thermal buckling behavior of functionally graded (FG) sandwiches plates using a reflected four-variable plate theory (RPT). The model developed accounts for mechanical and thermal interactions of the member without the need for a shear correction factor, preserving a realistic parabolic distribution of transverse shear stresses through the thickness of the sandwich plates.
The mechanical and thermal properties are assumed to vary continuously in the structural element based on volume fractions of the ceramic and metallic materials, using a power law distribution. The governing equations are created from the virtual work principle then solved analytically under Navier's method by assuming simply supported boundary conditions. The comparative results of the classical and advanced shear deformation theories show that the RPT is able to provide very accurate predictions of the thermal deflections and stresses while being less computationally intensive. The parametric studies show the change of material gradation, thickness ratios and temperature gradients on the critical buckling temperature of the FG sandwich structures. The results verify reliability when compared to 'classical theory', flexibility in modeling, and efficiency for applications of advanced thermo-mechanical design of FG sandwich-structures.

KEYWORDS

Functionally graded materials, sandwich plates, thermal buckling, shear deformation, power-law distribution, critical buckling temperature, thermo-mechanical behavior.

1. INTRODUCTION

This document describes and is written in accordance with author guidelines for the journals of the AIRCC series. It has been prepared in Microsoft Word as a .doc document. Other preparation methods are acceptable, but final, camera-ready versions must follow this layout. Microsoft Word terminology is used where appropriate throughout this document. Formatting instructions may seem overwhelming, but the easiest way to manage this is to use this template, adding headings and text accordingly.

Functionally graded materials (FGMs) are a new class of advanced composites with a continuous change in composition and structure through their thickness, resulting in smooth transitions in mechanical and thermal properties. They have been widely used in high-temperature environments like aerospace, nuclear, and automotive applications, where conventional laminated composites degrade due to delamination and interfacial stresses. FGMs combine ceramic and metallic materials into a smooth composition gradient, allowing for greater thermal resistance, strength-to-weight ratios, and durability under coupled thermal and mechanical loading.

In recent decades, the behavior of FG plates under thermal buckling is increasingly important as we demand reliable lightweight structures. The primary reasons for this include traditional plate theories, such as Classical Plate Theory (CPT) or First-Order Shear Deformation Theory (FSDT), which suffer from not correctly accounting for transverse shear deformation effects on thick plates. While Higher-Order Shear Deformation Plate Theories (HSDT) improve accuracy, they often introduce additional unknowns and complex formulations. This study presents a Refined Plate Theory (RPT) that contains only four unknowns and does not use a shear correction factor, providing a simple and accurate method for thermal buckling of FG sandwich plates.

2. PROBLEM FORMULATION

In a (x, y, z) rectangular coordinate system, consider a functionally graded sandwich plate which has a uniform thickness rectangular plate with three microscopically heterogeneous elastic layers as shown in Fig. 1 (Layer 1, Layer 2, and Layer 3). The top and bottom surface of the plate are located at the z = h/2 and z = -h/2, respectively, and the edges of the plate are parallel to the x- and y-axes. It is assumed that the volume fraction of the functionally graded layers changes continously through the thickness according to a power-law distribution.

$$V^{(1)} = \left(\frac{z - h_1}{h_2 - h_1}\right)^k, \quad z \in [h_1, h_2] \tag{1a}$$

$$V^{(2)} = 1, \quad z \in [h_2, h_3] \tag{1b}$$

$$V^{(3)} = \left(\frac{z - h_4}{h_3 - h_4}\right)^k, \quad z \in [h_3, h_4] \tag{1c}$$

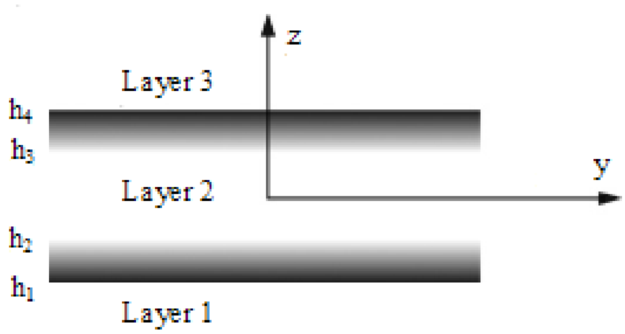

Figure 1. The material alternation through the thickness of the FG sandwich plate.

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3. HIGHER-ORDER PLATE THEORY

The displacements of a material point located at (x, y, z) in the plate may be written as
The displacements of a material point lying at (x, y, z) of the plate would be written as

$$u = u_0(x, y) - z\frac{\partial w_0}{\partial x} + \Psi(z)\,\theta_x \quad \text{(2a)}$$

$$v = v_0(x, y) - z\frac{\partial w_0}{\partial y} + \Psi(z)\,\theta_y \quad \text{(2b)}$$

$$w = w_0(x, y) \quad \text{(2c)}$$

in which, u, v, w are displacements in the x, y, z directions u_0, v_0 and w_0 are midplane displacements θ_x and θ_y and rotations of the yz and xz planes due to Buckling , respectively. $\Psi(z)$ represents shape a function that determines how the transverse shear strains and stresses are distributed across the thickness.

$$\Psi(z) = z\left(1 - \frac{4z^2}{3h^2}\right) \quad \text{(3)}$$

4. REFINED PLATE THEORY FOR FUNCTIONALLY GRADED PLATES

In contrast to the other theories, the present refined theory (RPT) only has four unknown functions while all other shear deformation theories have five (Houari et al.).

5. ASSUMPTIONS OF THE REFINED PLATE THEORY

The assumptions of the RPT are stated as follows:

(i) The displacements are small compared to the plate thickness and, therefore, strains thus considered to be infinitesimal.

(ii) The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of coordinates x, y only.

(iii) The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x and σ_y.

(iv) The displacements u in x-direction and v in y-direction consist of extension, Buckling , and shear components.

$$U = u_0 + u_b + u_s, \quad V = v_0 + v_b + v_s \quad \text{(4)}$$

6. KINEMATICS AND CONSTITUTIVE EQUATIONS

Again from the assumptions presented in the previous section, the field may be obtained using

$$\begin{cases} u(x,y,z) = u_0(x,y) - z\dfrac{\partial w_b}{\partial x} - f(z)\dfrac{\partial w_s}{\partial x} \\ v(x,y,z) = v_0(x,y) - z\dfrac{\partial w_b}{\partial y} - f(z)\dfrac{\partial w_s}{\partial y} \\ w(x,y,z) = w_b(x,y) + w_s(x,y) \end{cases} \quad (5)$$

where

$$f(z) = z - \Psi(z) \text{ and } \Psi(z) = z\left(1 - \frac{4z^2}{3h^2}\right) \quad (6)$$

The strains associated with the displacements in (6) are

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + z\,k_x^b + f(z)\,k_x^s \\ \varepsilon_y &= \varepsilon_y^0 + z\,k_y^b + f(z)\,k_y^s \\ \gamma_{xy} &= \gamma_{xy}^0 + z\,k_{xy}^b + f(z)\,k_{xy}^s \\ \gamma_{yz} &= g(z)\,\gamma_{yz}^s \\ \gamma_{xz} &= g(z)\,\gamma_{xz}^s \\ \varepsilon_z &= 0 \end{aligned} \quad (7)$$

7. GOVERNING EQUATIONS

The governing equations of equilibrium can be derived by way of the principle of virtual displacements. A strict application of the principle of virtual work in the present case yields

$$\int_{-h/2}^{h/2}\int_{\Omega}\begin{bmatrix} \sigma_x\delta\,\varepsilon_x + \sigma_y\delta\,\varepsilon_y + \tau_{xy}\delta\,\gamma_{xy} \\ +\tau_{yz}\delta\,\gamma_{yz} + \tau_{xz}\delta\,\gamma_{xz} \end{bmatrix} d\Omega\,dz - \int_{\Omega} q\delta\,w d\Omega = 0 \quad (8)$$

Where Ω are the top area of the plate.
Thus one can obtain the equilibrium equations associated with the present shear deformation theory,

$$\begin{aligned} &\delta\,u: \ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ &\delta\,v: \ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \\ &\delta\,w_b: \ \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} + q = 0 \\ &\delta\,w_s: \ \frac{\partial^2 M_x^s}{\partial x^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \frac{\partial^2 M_y^s}{\partial y^2} + \frac{\partial S_{xz}^s}{\partial x} + \frac{\partial S_{yz}^s}{\partial y} + q = 0 \end{aligned} \quad (9)$$

8. EXACT SOLUTION FOR A SIMPLY-SUPPORTED FGM SANDWICH PLATE

The following boundary conditions are imposed at the side edges for the present four variable refined plate theory:

$$v_0 = w_b = w_s = \frac{\partial w_s}{\partial y} = N_x = M_x^b = M_x^s = 0$$

at $x = -a/2,\ a/2$ (10)

$$u_0 = w_b = w_s = \frac{\partial w_s}{\partial x} = N_y = M_y^b = M_y^s = 0$$

at $y = -b/2,\ b/2$ (11)

To address this issue, Navier made the assumption that the shear mechanical and temperature loading parameters, q, T_1, T_2, and T_3 can be written as double trigonometric series of the form. The following provides the operator equation obtained,

$$[C]\{\Delta\} = \{P\} \tag{12}$$

where $\{\Delta\} = \{U, V, W_b, W_s\}^t$ and $[C]$ is the symmetric matrix given by

$$[C] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \tag{13}$$

in which:

$$\begin{aligned}
a_{11} &= A_{11}\lambda^2 + A_{66}\mu^2 \\
a_{12} &= \lambda\,\mu\,(A_{12} + A_{66}) \\
a_{13} &= -\lambda\,[B_{11}\lambda^2 + (B_{12} + 2B_{66})\,\mu^2] \\
a_{14} &= -\lambda\,[B_{11}^s\lambda^2 + (B_{12}^s + 2B_{66}^s)\,\mu^2] \\
a_{22} &= A_{66}\lambda^2 + A_{22}\mu^2 \\
a_{23} &= -\mu\,[(B_{12} + 2B_{66})\,\lambda^2 + B_{22}\mu^2] \\
a_{24} &= -\mu\,[(B_{12}^s + 2B_{66}^s)\,\lambda^2 + B_{22}^s\mu^2] \\
a_{33} &= D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4 \\
a_{34} &= D_{11}^s\lambda^4 + 2(D_{12}^s + 2D_{66}^s)\lambda^2\,\mu^2 + D_{22}^s\,\mu^4 \\
a_{44} &= H_{11}^s\lambda^4 + 2(H_{11}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 \\
&\quad - A_{55}^s\lambda^2 - A_{44}^s\mu^2
\end{aligned} \tag{14}$$

9. NUMERICAL RESULTS

The four variable refined plate theory for FG sandwich plates has been introduced in this study, and compared to classic solutions using other shear deformation theories found in the literature. Symmetric and non-symmetric sandwich plates have been considered. One can assume that the core of the sandwich plate is completely ceramic and the bottom and top surfaces of the sandwich plates are ceramic rich. In addition, it is noted that there are different types of sandwich plates in use: Next, we conduct a thermomechanical Buckling analysis for metal-ceramic combinations. The material combination will be Aluminum and Alumina. The material properties of Aluminum and Alumina are as follows:

- Metal (Aluminium, Al): $E_M = 70 \times 10^9$ N/m2; $\nu = 0.3$; $\alpha_m = 23 \times (10^{-6} / °C)$.
- Ceramic (Alumina, Al2O3): $E_C = 380 \times 10^9$ N/m2; $\nu = 0.3$; $\alpha_c = 7.4 \times (10^{-6} / °C)$.

The various non-dimensional parameters used are

- Center deflection $\bar{w} = \dfrac{10^3}{q_0 a^4 / (E_0 h^3) + (10^3 \alpha_0 \bar{T} 2 a^2)/h} w\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$,
- axial stress $\bar{\sigma}_x = \dfrac{10}{q_0 a^2 / (h^2) + (10 E_0 \alpha_0 \bar{T} 2 a^2)/h^2} \sigma_x\left(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{h}{2}\right)$,
- shear stress $\bar{\tau}_{xz} = \dfrac{1}{q_0 a / (h) + (E_0 \alpha_0 \bar{T} 2 a)/10h} \tau_{xz}\left(0, \dfrac{b}{2}, 0\right)$,
- thickness coordinate $\bar{z} = z / h$.

where the reference value is taken as E_0= 1 GPa and $\alpha_0 = 10^{-6}$ /°C . We also take the shear correction factor K = 5/6 in FSDPT. Numerical results are presented using different plate theories.

Table 1 shows the dimensionless center deflection $\bar{w}$ for an FG sandwich plate subjected to mechanical and thermal loads. The deflections are considered for k= 0, 1, 2, 3, 4, and 5 and different types of FG sandwich plates. It shows that the effect of shear deformation is to increase the deflection. For a FG plate, the deflections increase as the core thickness decreases whereas k increases.

TABLE 1. Comparison of nondimensional center deflections $\bar{w}$ for different FG sandwich square.

Mode	**Theory**	**1–0–1**	**3–1–3**	**2–1–2**	**1–1–1**
0	**Theory**	-2.462315	-2.462318	-2.462321	-2.462329
0	Present	-2.389751	-2.389756	-2.389760	-2.389768
1	SSDPT	-2.475286	-2.563974	-2.608923	-2.732847
1	Present	-2.408192	-2.496037	-2.539824	-2.661975
2	SSDPT	-2.182506	-2.265432	-2.311025	-2.450867
2	Present	-2.119845	-2.202663	-2.246392	-2.384919
3	SSDPT	-2.083972	-2.151864	-2.193271	-2.330915
3	Present	-2.021634	-2.089275	-2.129042	-2.263951
4	SSDPT	-2.040318	-2.096728	-2.134587	-2.266743
4	Present	-1.979865	-2.035624	-2.071893	-2.201675
5	SSDPT	-2.020857	-2.068412	-2.102935	-2.228654
5	Present	-1.959284	-2.006812	-2.040456	-2.165102

Tables 2 list, the axial stress $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{xz}$ for k = 0, 1, 2, 3, 4, and 5 and different types of sandwich plates. It can be seen that both the axial stress $\bar{\sigma}_x$ and the shear stress $\bar{\tau}_{xz}$ decrease ;

Figure 2 shows the effects of the aspect ratio a/b on the dimensionless deflection $\bar{w}$ of symmetric and nonsymmetric FG plate ($k = 1.5$).

Table 2. Comparison of nondimensional axial stress deflections $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{xz}$ for different FG sandwich square plates.

Mode	**Theory**	**1–0–1**	**3–1–3**	**2–1–2**	**1–0–1**
0	Present	0.808325	0.808512	0.808476	0.174829
0	SSDPT	0.796942	0.797131	0.797095	0.171952
1	Present	1.078052	1.059874	1.051136	0.272845
1	SSDPT	1.063184	1.045237	1.036514	0.277493
2	Present	1.137584	1.120836	1.111724	0.271248
2	SSDPT	1.121943	1.105327	1.096285	0.272914
3	Present	1.157962	1.144132	1.135719	0.270394
3	SSDPT	1.141923	1.128312	1.120045	0.269823
4	Present	1.158142	1.144326	1.135927	0.272016
4	SSDPT	1.141874	1.128191	1.119865	0.270284
5	Present	1.171002	1.161213	1.154283	0.274918
5	SSDPT	1.154681	1.145084	1.138256	0.272453

The displacement induced by employing the First-order Shear Deformation Plate Theory (FSDPT) is determined to be the largest numerical value and the displacement induced by employing other theories to be the smallest numerical value. It is observed that displacement, as a result of utilizing various shear deformation theories, decreases with an increased aspect ratio. It should be noted that the displacement induced by using the presently introduced theory and Timoshenko shear deformation plate theory (TSDPT) are close values in relation to each other and the values given by each of these methods takes, in the center, an intermediate value between the FSDPT and classical plate theory (CPT). It can be determined that CPT does yields a low amount for plate deflection.

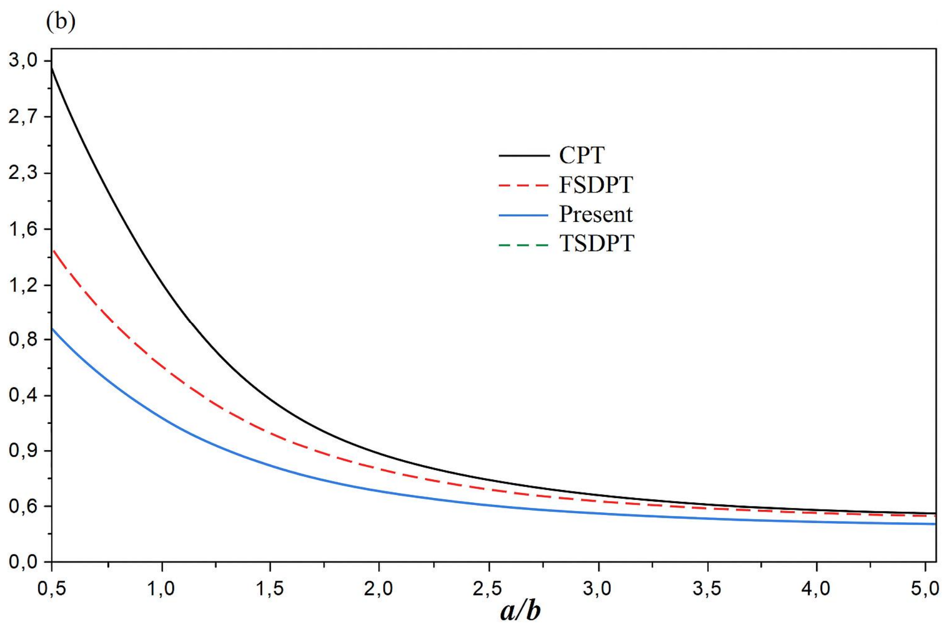

Figure 2. Effect of the aspect ratio a/b on the dimensionless center deflection $\bar{w}$ of nonsymmetric FG sandwich plates (2-2-1), ($k = 1.5$) by applying different shear deformation theories.

Figures 3 and 4 contain the plots of the axial stress $\bar{\sigma}_x$ and the shear stress $\bar{\tau}_{xz}$ through-the-thickness of both symmetric and nonsymmetric FG plate ($k = 1.5$) using various shear deformation theories. It is also noted that the present theory and TSDPT are coincided.

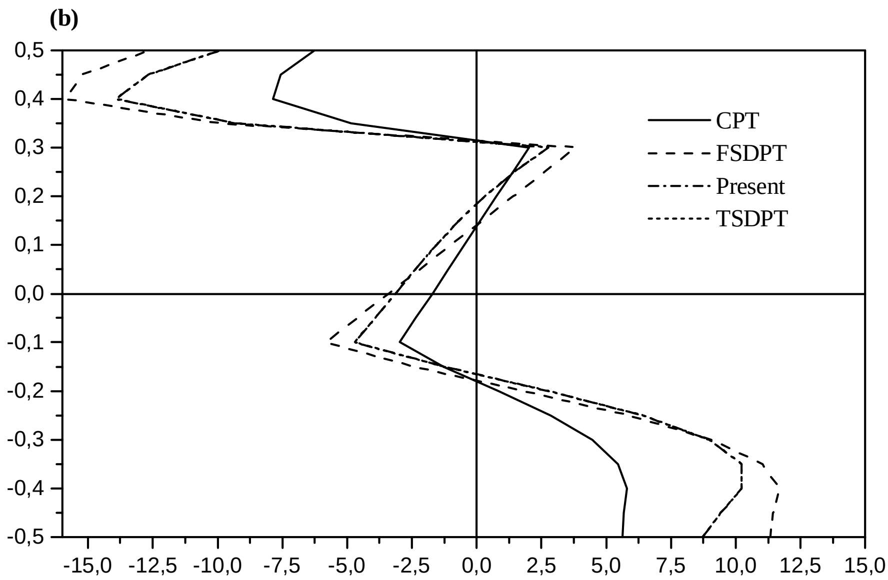

Figure 3. Comparison of the variation of nondimensional axial stress $\bar{\sigma}_x$ across the thickness of nonsymmetric FG sandwich plates (2-2-1), ($k = 1.5$) by applying different shear deformation theories.

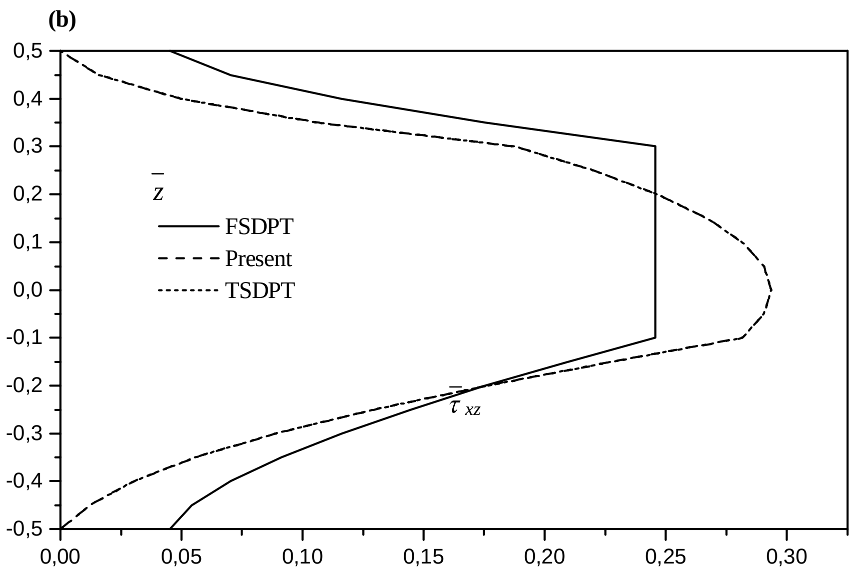

Figure 4. Comparison of the variation of nondimensional shear stress $\bar{\tau}_{xz}$ across the thickness of nonsymmetric FG sandwich plates (2-2-1), ($k = 1.5$) by applying different shear deformation theories.

3. CONCLUSIONS

This paper develops a new four variable refined plate theory (RPT) to study for the first time the thermomechanical response of simply supported FG sandwich plates. Unlike any other theory, the presented theory leads to four governing equations which results in significantly less computational effort compared to the other higher-order theories available in the literature, in which the number of governing equations is greater. The results are also compared to the shear deformation theories, and in general, the fully ceramic plates produce the lowest deflections and the lowest transverse shear stress. The role of the gradient in the material properties is essential in assessing the response of the FGM plates. The compositions of the ceramic and metal with a continuously varying volume fraction eliminate interface problems of sandwich plates, providing smooth stresses distributions. In all comparison studies the deflections and stresses obtained using the present refinements theory (with four unknowns) and the other five higher-order shear deformation theories (five unknowns) are almost identical. Thus, it can be stated that the presented theory RPT is accurate and simple in analyzing the thermomechanical buckling response of FG plates.

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Authors

Dr. Ahmed HAMIDI is a researcher at the Eco-Materials: Innovations and Applications (E.M.I.A.) Laboratory, Tahri Mohamed University of Béchar, Algeria. He obtained his Ph.D. in Civil Engineering from Djillali Liabes University, Sidi Bel Abbes, in 2015, and his Habilitation in 2019. His research focuses on functionally graded and composite materials, thermo-mechanical behavior of advanced structures, and refined plate and shell theories. He has published numerous papers in international journals and conferences, contributing significantly to the modeling and analysis of FGM and nanostructural systems.

Short Biography